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PROPERTIES OF EIGENFUNCTION EXPANSION FORM FOR THE RIGID LINE PROBLEM IN DISSIMILAR MEDIA

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Abstract— In this paper, the eigenfunction expansion form (abbreviated as EEF) in the rigid line problem in dissimilar media is derived. The properties of the EEF are discussed in detail. After using Betti's reciprocal theorem for a particular contour, several path independent integrals are obtained. All the coefficients in the EEF, including the K_{1R} and K_{2R} values can be related to the corresponding path independent integrals. It is found that the J-integral takes a definitely negative value in the present case. A possibility for formulating the weight function is also suggested. Finally, a boundary value problem for a single rigid line embedded in dissimilar media is studied and solved.

1. INTRODUCTION

The opposite of a crack, in a certain sense, is a cut in the material that is filled with a rigid lemella. There is no uniform terminology in this aspect; in the plane elasticity case we shall call it the rigid line for brevity. Contrary to the crack, the rigid line transmits tractions, but prevents a displacement discontinuity. There is a considerable amount of literature on this topic. A fairly complete list of references can be found (Brussat and Westmann, 1975; Chen, 1986; Chen and Hasebe, 1992; Dundurs and Markenscoff, 1989; England, 1971; Erdogan and Gupta, 1972; Hasebe and Takeuchi, 1985; Hasebe *et al.*, 1984; Wang *et al.*, 1985).

The characteristics of the stress field near the tip of a rigid line can be found from an earlier investigation (England, 1971). After analyzing the behaviour of the stresses in the vicinity of a rigid line tip, the leading term of the stress components was obtained (Hasebe *et al.*, 1984). The coefficient involved in the leading term is defined as the stress singularity coefficient (Hasebe 1985; Wang *et al.*, 1985), which in turn depends on the loading condition and rigid line geometry.

Here and after, we call K_{1R} and K_{2R} the stress singularity coefficients (abbreviated SSC). In the isotropic case, the SSC can be defined by

$$K_{1R} - iK_{2R} = \lim_{z \to 0} 2\sqrt{2\pi z}\,\phi'(z) \tag{1}$$

where $\phi'(z)$ is the complex potential used by Muskhelishvili (1953).

In this paper, the rigid line is embedded in the dissimilar media with the elastic constants G_1, κ_1, ν_1, for the upper plane, and G_2, κ_2, ν_2 for the lower plane, respectively

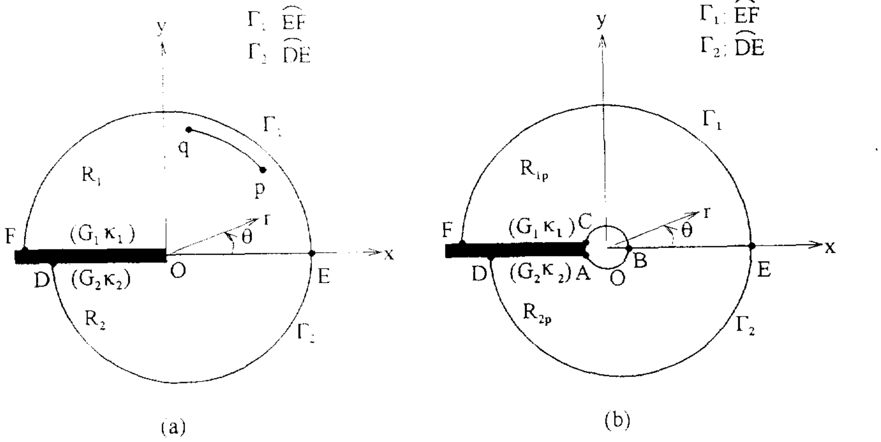

Fig. 1. Region near a rigid line tip (a) for a physical problem: (b) for defining the eigenfunction expansion form including the negative eigenvalues.

[Fig. 1(a)]. The relevant EEF is derived by the complex variable method (Muskhelishvili, 1953). The obtained result can be considered as a counterpart of that obtained in the interface crack problem (Rice, 1988). Similar to the crack problem, the eigenvalues consist of two kinds. One is $n/2-i\varepsilon$ and the other is n (n-integer). The EEF with positive real and negative real eigenvalues is presented in this paper.

Similar to the interface crack problem, we propose the following definition for the SSC in the dissimilar material case

$$K_{1R}-\mathrm{i}K_{2R} = 2\sqrt{2\pi}\sqrt{\frac{G_2\kappa_1}{G_1\kappa_2}}\, e^{\pi\varepsilon} \lim_{z\to 0} z^{1/2-i\varepsilon}\phi'(z) \tag{2}$$

where $\varepsilon = \{\log[(\kappa_2(G_1+G_2\kappa_1))/(\kappa_1(G_2+G_1\kappa_2))]\}/(2\pi)$.

A work-line integral is introduced and discussed, which is formulated by a subtraction of two works along a curve. One is obtained from the work done by the traction of the α-field to the displacement of the β-field, and other is obtained from the work done by the traction of the β-field to the displacement of the α-field. The path of integration is chosen around the rigid line tip, and the α- and β-fields are one term of the EEF. In most cases, a pseudo-orthogonal property of the EEF has been found. This is to say, only some particular pairs of the EEF have a contribution to the work-like integral. Otherwise, the integral is equal to zero. It is proved that the J-integral in the rigid line case takes definitely negative value.

2. PROPERTIES OF THE EIGENFUNCTION EXPANSION FORM

The eigenfunction expansion form in the interface rigid line problem will be derived by using the complex variable function method (Muskhelishvili, 1953). According to this method, the stresses $(\sigma_x, \sigma_y, \sigma_{xy})$, the resultant forces (X, Y) and the displacements (u, v) can be derived by two complex potentials $\phi(z)$ and $\omega(z)$ (or the pair $\phi(z)$ and $\psi(z)$)

$$\sigma_x+\sigma_y = 4\,\mathrm{Re}\,[\Phi(z)]$$

$$\sigma_y-\mathrm{i}\sigma_{xy} = \Phi(z)+(z-\bar{z})\overline{\Phi'(z)}+\overline{\Omega(z)} = \Phi(z)+\overline{\Phi(z)}+z\overline{\Phi'(z)}+\overline{\Psi(z)} \tag{3}$$

$$P = -Y+\mathrm{i}X = \phi(z)+(z-\bar{z})\overline{\phi'(z)}+\overline{\omega(z)} = \phi(z)+z\overline{\phi'(z)}+\overline{\psi(z)} \tag{4}$$

$$2G(u+\mathrm{i}v) = \kappa\phi(z)-(z-\bar{z})\overline{\phi'(z)}-\overline{\omega(z)} = \kappa\phi(z)-z\overline{\phi'(z)}-\overline{\psi(z)} \tag{5}$$

where $\Phi(z) = \phi'(z)$, $\Omega(z) = \omega'(z)$, $\Psi(z) = \psi'(z)$ and $\omega(z) = z\phi'(z)+\psi(z)$. G is the shear modulus of elasticity, $\kappa = 3-4\nu$ for the plane strain problem, $\kappa = (3-\nu)/(1+\nu)$ for the plane stress problem, and ν is the Poisson's ratio.

We seek the solution in the region $R(R = R_1 + R_2$, Fig. 1(a)) where a rigid line is embedded. The elastic constants and the complex potentials are denoted by G_1, κ_1, $\phi_1(z)$, $\omega_1(z)$, $\psi_1(z)$ and G_2, κ_2, $\phi_2(z)$, $\omega_2(z)$, $\psi_2(z)$ for the upper and lower planes, respectively. From eqns (4) and (5), the continuation condition of the resultant force and displacement gives rise to the following relations:

$$\phi_1'(x)+\omega_1'(x) = \phi_2'(x)+\omega_2'(x) \quad (x>0) \tag{6}$$

$$G_2(\kappa_1\phi_1'(x)-\omega_1'(x)) = G_1(\kappa_2\phi_2'(x)-\omega_2'(x)) \quad (x>0). \tag{7}$$

Generally, we assume the rigid line to be fixed. Thus, the condition for the adjacent bonded media is obtainable

$$\kappa_1\phi_1'(x)-\omega_1'(x) = 0 \quad (x<0) \tag{8}$$

$$\kappa_2\phi_2'(x)-\omega_2'(x) = 0 \quad (x<0). \tag{9}$$

By using the available result in the interface crack problem (Rice, 1988; Rice and Sih, 1965), we can directly investigate the EEF in the form

$$\phi_1(z) = p_1 z^{a+ib} \quad (z\in R_1 \text{ or } R_{1p}) \tag{10}$$

$$\omega_1(z) = q_1 z^{a+ib} \quad (z\in R_1 \text{ or } R_{1p}) \tag{11}$$

$$\phi_2(z) = p_2 z^{a+ib} \quad (z\in R_2 \text{ or } R_{2p}) \tag{12}$$

$$\omega_2(z) = q_2 z^{a+ib} \quad (z\in R_2 \text{ or } R_{2p}) \tag{13}$$

where a and b are two real values and by cancelling a small region from $R_1(R_2)$, we get the region $R_{1p}(R_{2p})$, respectively [Fig. 1(b)].

Note that, in the derivation, the following definition is useful

$$z^{a+ib} = e^{(a+ib)\log z} = e^{(a\log r - b\theta)+i(a\theta + b\log r)} \quad (\text{with } z = r e^{i\theta}) \tag{14}$$

Substituting eqns (10), (11), (12) and (13) into the eqns (6), (7), (8) and (9), we obtain

$$\begin{vmatrix} 1 & 1 & -1 & -1 \\ G_2\kappa_1 & -G_2 & -G_1\kappa_2 & G_1 \\ \kappa_1 s & 1 & 0 & 0 \\ 0 & 0 & \kappa_2 & -s \end{vmatrix} \begin{vmatrix} p_1 \\ q_1 \\ p_2 \\ q_2 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \end{vmatrix} \tag{15}$$

where

$$s = e^{2\pi i(a+ib)} \tag{16}$$

The non-trivial solution condition of eqn (15) leads to

$$(s-1)[\kappa_1(G_2+G_1\kappa_2)s+\kappa_2(G_1+G_2\kappa_1)] = 0. \tag{17}$$

The solutions of eqn (17) can be devided into two groups. The eigenvalue in the first group can be expressed by

$$s = 1 \tag{18}$$

$$a - \mathrm{i}b = n \quad (n\text{-integer}). \tag{19}$$

In this case, the eigenvalue takes real 1. Furthermore, the non-trivial solution is obtainable

$$\phi_1(z) = \sqrt{\frac{1+\kappa_2}{1+\kappa_1}}\, p z^n \quad (z \in R_1 \text{ or } R_{1p}) \tag{20}$$

$$\omega_1(z) = \kappa_1 \sqrt{\frac{1+\kappa_2}{1+\kappa_1}}\, \bar{p} z^n \quad (z \in R_1 \text{ or } R_{1p}) \tag{21}$$

$$\phi_2(z) = \sqrt{\frac{1+\kappa_1}{1+\kappa_2}}\, p z^n \quad (z \in R_2 \text{ or } R_{2p}) \tag{22}$$

$$\omega_2(z) = \kappa_2 \sqrt{\frac{1+\kappa_1}{1+\kappa_2}}\, \bar{p} z^n \quad (z \in R_2 \text{ or } R_{2p}) \tag{23}$$

where p is an arbitrary complex constant.

The eigenvalue in the second group can be expressed by

$$s = -\frac{\kappa_2(G_1 + G_2\kappa_1)}{\kappa_1(G_2 + G_1\kappa_2)} \tag{24}$$

$$a - \mathrm{i}b = n + \tfrac{1}{2} - \mathrm{i}\varepsilon \quad (n\text{-integer}) \tag{25}$$

where

$$\varepsilon = \frac{1}{2\pi} \log\left[\frac{\kappa_2(G_1 + G_2\kappa_1)}{\kappa_1(G_2 + G_1\kappa_2)}\right]. \tag{26}$$

It is of interest to point out that, if the following condition.

$$\frac{\kappa_2(G_1 + G_2\kappa_1)}{\kappa_1(G_2 + G_1\kappa_2)} = 1, \quad \text{or} \quad \frac{G_1}{G_2} = \frac{\kappa_1(\kappa_2 - 1)}{\kappa_2(\kappa_1 - 1)} \tag{27}$$

is satisfied, then $\varepsilon = 0$. That is to say in a particular condition which is shown by eqn (27), the oscillating singularity vanishes in the interface rigid line problem. Clearly, there is no counterpart in the interface crack problem. To simply analysis, we shall exclude this particular solution. Similarly, we can obtain the following nontrivial solution.

$$\phi_1(z) = \sqrt{\frac{G_1\kappa_2}{G_2\kappa_1}}\, \mathrm{e}^{-\pi\varepsilon} p z^{n+1/2-\mathrm{i}\varepsilon} \quad (z \in R_1 \text{ or } R_{1p}) \tag{28}$$

$$\omega_1(z) = -\kappa_1 \sqrt{\frac{G_1\kappa_2}{G_2\kappa_1}}\, \mathrm{e}^{\pi\varepsilon} \bar{p} z^{n+1/2+\mathrm{i}\varepsilon} \quad (z \in R_1 \text{ or } R_{1p}) \tag{29}$$

$$\phi_2(z) = \sqrt{\frac{G_2\kappa_1}{G_1\kappa_2}}\, \mathrm{e}^{\pi\varepsilon} p z^{n+1/2-\mathrm{i}\varepsilon} \quad (z \in R_2 \text{ or } R_{2P}) \tag{30}$$

$$\omega_2(z) = -\kappa_2\sqrt{\frac{G_2\kappa_1}{G_1\kappa_2}}\,e^{-\pi\varepsilon}\bar{p}z^{n+1/2+i\varepsilon} \quad (z \in R_2 \text{ or } R_{2P}) \tag{31}$$

where, as before, p is an arbitrary complex constant.

Finally, the EEF is formulated by the linear combination of the above-mentioned non-trivial solutions, and it takes

$$\phi_1(z) = e_1 z^{1/2-i\varepsilon}f(z) + g_1 g(z) \quad (z \in R_1 \text{ or } R_{1P}) \tag{32}$$

$$\omega_1(z) = f_1 z^{1/2+i\varepsilon}\bar{f}(z) + h_1\bar{g}(z) \quad (z \in R_1 \text{ or } R_{1P}) \tag{33}$$

$$\phi_2(z) = e_2 z^{-1/2-i\varepsilon}f(z) + g_2 g(z) \quad (z \in R_2 \text{ or } R_{2P}) \tag{34}$$

$$\omega_2(z) = f_2 z^{1/2+i\varepsilon}\bar{f}(z) + h_2\bar{g}(z) \quad (z \in R_2 \text{ or } R_{2P}) \tag{35}$$

where

$$e_1 = \sqrt{\frac{G_1\kappa_2}{G_2\kappa_1}}\,e^{-\pi\varepsilon} \qquad f_1 = -\kappa_1\sqrt{\frac{G_1\kappa_2}{G_2\kappa_1}}\,e^{\pi\varepsilon}$$

$$e_2 = \sqrt{\frac{G_2\kappa_1}{G_1\kappa_2}}\,e^{\pi\varepsilon} \qquad f_2 = -\kappa_2\sqrt{\frac{G_2\kappa_1}{G_1\kappa_2}}\,e^{-\pi\varepsilon}$$

$$g_1 = \sqrt{\frac{1+\kappa_2}{1-\kappa_1}} \qquad h_1 = \kappa_1\sqrt{\frac{1+\kappa_2}{1+\kappa_1}}$$

$$g_2 = \sqrt{\frac{1-\kappa_1}{1-\kappa_2}} \qquad h_2 = \kappa_2\sqrt{\frac{1+\kappa_1}{1+\kappa_2}} \tag{36}$$

$$\varepsilon = \frac{1}{2\pi}\log\left(\frac{\kappa_2(G_1+G_2\kappa_1)}{\kappa_1(G_2+G_1\kappa_2)}\right) \tag{37}$$

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \tag{38}$$

$$g(z) = \sum_{n=-\infty}^{\infty} c_n z^n \tag{39}$$

where a_n and c_n are complex coefficients.

It is easy to see the obtained EEF contains two parts. The first one is composed of the complex eigenvalue, and the second one is composed of the integer eigenvalue. Differing from the EEF suggested previously, the terms with the negative values of n in $f(z)$ and $g(z)$ are also included in summation. The displacements must be finite for the bonded materials, and thus, in eqns (38) and (39), we can conclude:

(a) The terms corresponding to $n > 0$ for both functions $f(z)$ and $g(z)$ are physically possible in the region $R[R = R_1 + R_2$, Fig. 1(a)], and the term of $n = 0$ in $g(z)$ represents a rigid motion of body.

(b) The terms corresponding to $n < 0$ for both functions $f(z)$ and $g(z)$ are only physically possible in the region $R_P[R_P = R_{1P} + R_{2P}$, Fig. 1(b)], where a small region with contour ABC has been excluded. We shall soon prove that the terms corresponding to $n < 0$ have many uses in the following analysis.

Now the first property of EEF can be reached as follows. If the displacements u and v are derived from one term (with the eigenvalue $n+1/2-i\varepsilon$ for the first part, or n for the second part) of EEF in eqns (32)–(35), then $u_* = \partial u/\partial x$ and $v_* = \partial v/\partial x$ is also a term (with the eigenvalue $n-1/2-i\varepsilon$ for the first part, or $n-1$ for the second part) of EEF in eqns (32)–(35). The exception is the case of $n = 0$ in the second part of EEF.

The following proof is carried out for a term of the first part of EEF defined in upper half-plane. In fact, from the above formulation we have

$$2G_1(u+iv) = \kappa_1\phi_1(z)-(z-\bar{z})\overline{\phi_1'(\bar{z})}-\overline{\omega_1(\bar{z})} \quad (z\in R_{1p}) \tag{40}$$

where

$$\begin{aligned}\phi_1(z) &= e_1 a_n z^{n+1/2-i\varepsilon} \quad (z\in R_{1p})\\ \omega_1(z) &= f_1 \bar{a}_n z^{n+1/2+i\varepsilon} \quad (z\in R_{1p}).\end{aligned} \tag{41}$$

Then we can get

$$2G_1(u_*+iv_*) = 2G_1\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right) = \kappa_1\phi_1'(z)-(z-\bar{z})\overline{\phi_1''(\bar{z})}-\overline{\omega_1'(\bar{z})} \tag{42}$$

and rewrite the above equation in the form

$$2G_1(u_*+iv_*) = \kappa_1\phi_{1*}(z)-(z-\bar{z})\overline{\phi_{1*}'(\bar{z})}-\overline{\omega_{1*}(\bar{z})} \tag{43}$$

where

$$\begin{aligned}\phi_{1*}(z) &= \phi_1'(z) = e_1(n+\tfrac{1}{2}-i\varepsilon)a_n z^{n-1/2-i\varepsilon}\\ \omega_{1*}(z) &= \omega_1'(z) = f_1(n+\tfrac{1}{2}+i\varepsilon)\bar{a}_n z^{n-1/2+i\varepsilon}.\end{aligned} \tag{44}$$

Clearly, the above complex potential belongs to one term of the first part of EEF with the eigenvalue $n-1/2-i\varepsilon$. Similar proof can be performed for the case of lower half-plane. Also, it is easy to prove the same property for the case of the second part of EEF.

Before discussing the second property of EEF, the following result obtained by Blueckner (1973) is useful. If there are two cases of deformation state in plane elasticity, namely, $\phi_{(\alpha)}(z)$, $\omega_{(\alpha)}(z)$ and $\phi_{(\beta)}(z)$, $\omega_{(\beta)}(z)$, respectively, then the following integral can be defined:

$$2GW_{pq} = 2G\int_{pq}(u_{i(\alpha)}\sigma_{ij(\beta)}-u_{i(\beta)}\sigma_{ij(\alpha)})n_j\,ds \tag{45}$$

where p and q are two points in the elastic plane [Fig. 1(a)]. Clearly, by the use of Betti's reciprocal theorem (Sokolnikoff, 1956), the above integral is a path independent integral. After some manipulation, the above integral can be evaluated by the following equation:

$$2GW_{pq} = (\kappa+1)\,\mathrm{Im}\,H(z)|_p^q+\mathrm{Im}\,R|_p^q \tag{46}$$

where

$$\begin{aligned}H(z) &= \int h(z)\,dz\\ h(z) &= -\omega_\alpha(z)\phi_\beta'(z)+\omega_\beta(z)\phi_\alpha'(z)\end{aligned} \tag{47}$$

$$R = -\kappa P_\alpha \overline{P_\beta} + (\kappa+1)(P_\alpha - \phi_\alpha)(\overline{P_\beta} - \overline{\phi_\beta}) \tag{48}$$

and P_α and P_β are the resultant force functions derived from the α- and the β-stress fields, respectively.

For convenience to get the derivation cited below, the following definition is used. A particular stress field is called the eigen-stress field (abbreviated as ESF) in this paper. It is defined in a dissimilar bonded region as shown in Fig. 1(b) and satisfies: (a) all the governing equations of plane elasticity; (b) the displacement and traction continuations along the bonded line ($x > 0, y = 0$); (c) the fixed condition along the upper and lower faces of the rigid line. Clearly, for any pairs of the α- and the β-stress fields which belong to ESF, the following path independent integral is obtainable:

$$W = \int_{\Gamma_2 - \Gamma_1} (u_{i(\alpha)}\sigma_{ij(\beta)} - u_{i(\beta)}\sigma_{ij(\alpha)}) n_j \, ds \tag{49}$$

where Γ_2 is any integration path starting at any point D of the lower rigid line face and Γ_1 is any integration path ended at any point F of the upper rigid line face [Fig. 1(b)]. Obviously, each term in the EEF can serve as ESF mentioned above. In the following, three types of integral (49) will be investigated.

(a) In the first case, both the α- and the β-stress fields are taken from the first part of EEF in eqns (32)–(35). It is assumed that the α-stress field $(u_{i(\alpha)}, \sigma_{ij(\alpha)})$ is derived from the following complex potentials:

$$\begin{aligned} \phi_1(z) &= e_1 a_n z^{n+1/2-i\varepsilon} \quad (z \in R_{1p}) \\ \omega_1(z) &= f_1 \overline{a_n} z^{n+1/2+i\varepsilon} \quad (z \in R_{1p}) \\ \phi_2(z) &= e_2 a_n z^{n+1/2-i\varepsilon} \quad (z \in R_{2p}) \\ \omega_2(z) &= f_2 \overline{a_n} z^{n+1/2+i\varepsilon} \quad (z \in R_{2p}) \end{aligned} \tag{50}$$

and the β-stress field $(u_{i(\beta)}, \sigma_{ij(\beta)})$ is derived from the following complex potentials:

$$\left.\begin{aligned} \phi_1(z) &= e_1 b_m z^{m+1/2-i\varepsilon} \quad (z \in R_{1p}) \\ \omega_1(z) &= f_1 \overline{b_m} z^{m+1/2+i\varepsilon} \quad (z \in R_{1p}) \\ \phi_2(z) &= e_2 b_m z^{m+1/2-i\varepsilon} \quad (z \in R_{2p}) \\ \omega_2(z) &= f_2 \overline{b_m} z^{m+1/2+i\varepsilon} \quad (z \in R_{2p}) \end{aligned}\right] . \tag{51}$$

After using eqns (45)–(48), we can get

$$W = \int_{\Gamma_2 - \Gamma_1} (u_{i(\alpha)}\sigma_{ij(\beta)} - u_{i(\beta)}\sigma_{ij(\alpha)}) n_j \, ds = \begin{cases} 0 & \text{if } n+m+1 \neq 0 \\ H_1 & \text{if } n+m+1 = 0 \end{cases} \tag{52}$$

where

$$H_1 = -\pi\left(\frac{\kappa_1(1+\kappa_2)}{G_1} + \frac{\kappa_2(1+\kappa_1)}{G_2}\right)\mathrm{Re}\,[a_n b_m (n + \tfrac{1}{2} - i\varepsilon)]. \tag{53}$$

(b) In the second case, both α- and the β-stress fields are taken from the second part of the EEF in eqns (32)–(35). It is assumed that the α-stress field $(u_{i(\alpha)}, \sigma_{ij(\alpha)})$ is derived from the following complex potentials:

$$\begin{aligned} \phi_1(z) &= g_1 c_n z^n \quad (z \in R_{1p}) \\ \omega_1(z) &= h_1 c_n z^n \quad (z \in R_{1p}) \\ \phi_2(z) &= g_2 c_n z^n \quad (z \in R_{2p}) \\ \omega_2(z) &= h_2 \bar{c}_n z^n \quad (z \in R_{2p}) \end{aligned} \tag{54}$$

and the β-stress field $(u_{i(\beta)}, \sigma_{ij(\beta)})$ is derived from the following complex potentials

$$\begin{aligned} \phi_1(z) &= g_1 d_m z^m \quad (z \in R_{1p}) \\ \omega_1(z) &= h_1 \bar{d}_m z^m \quad (z \in R_{1p}) \\ \phi_2(z) &= g_2 d_m z^m \quad (z \in R_{2p}) \\ \omega_2(z) &= h_2 \bar{d}_m z^m \quad (z \in R_{2p}). \end{aligned} \tag{55}$$

After using eqns (45)–(48) we can get

$$W = \int_{\Gamma_2 - \Gamma_1} (u_{i(\alpha)}\sigma_{ij(\beta)} - u_{i(\beta)}\sigma_{ij(\alpha)})n_j \, \mathrm{d}s = \begin{cases} 0 & \text{if} \quad n+m \neq 0 \\ H_2 & \text{if} \quad n+m = 0 \end{cases} \tag{56}$$

where

$$H_2 = n\pi\left(\frac{\kappa_1(1+\kappa_2)}{G} + \frac{\kappa_2(1+\kappa_1)}{G_2}\right) \mathrm{Re}\,[(c_n \bar{d}_m)]. \tag{57}$$

(c) In the third case, the α-stress field is taken from the first part of the EEF in eqns (32)–(35) and the β-stress field is taken from the second part of the EEF in eqns (32)–(35). It is assumed that the α-stress field $(u_{i(\alpha)}, \sigma_{ij(\alpha)})$ is derived from the complex potentials shown by eqn (50), and the β-stress field $(u_{i(\beta)}, \sigma_{ij(\beta)})$ is derived from the complex potentials shown by eqn (55). After using eqns (45)–(48), in any case of n and m in eqns (50) and (55) we can get

$$W = \int_{\Gamma_2 - \Gamma_1} (u_{i(\alpha)}\sigma_{ij(\beta)} - u_{i(\beta)}\sigma_{ij(\alpha)})n_j \, \mathrm{d}s = 0. \tag{58}$$

From eqns (52), (56) and (58) we see that, only some particular pairs in the EEF have a contribution to the path independent integral (49).

3. PATH INDEPENDENT INTEGRALS

As mentioned above, if some pairs of the stress fields which belong to ESF are taken in eqn (49), the relevant path independent integrals can be obtained. Several important cases are cited below.

(a) We take the α-stress field $(u_{i(\alpha)} = u_i, \sigma_{ij(\alpha)} = \sigma_{ij})$ as a physical stress field caused by some tractions acting on the boundary of the dissimilar body. Therefore, the corresponding complex potential takes the form shown by eqns (32)–(35) and the functions $f(z)$ and $g(z)$ will be

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

$$g(z) = \sum_{n=0}^{\infty} c_n z^n. \tag{59}$$

By the use of the definition shown by eqn (2), we can obtain the relation

$$K_{1R} - \mathrm{i}K_{2R} = 2\sqrt{2\pi}\sqrt{\frac{G_2\kappa_1}{G_1\kappa_2}}\,\mathrm{e}^{\pi\varepsilon}\lim_{z\to 0} z^{1/2-\mathrm{i}\varepsilon}\phi_1'(z) = 2\sqrt{2\pi}(\tfrac{1}{2}-\mathrm{i}\varepsilon)a_0. \tag{60}$$

In addition, we choose the following β-stress field : $u_{i(\beta)} = \partial u_i/\partial x$ and $\sigma_{ij(\beta)} = \partial\sigma_{ij}/\partial x$. Using the properties of EEF mentioned above, it is easy to see that only one pair, which consists of one term with the eigenvalue $1/2-\mathrm{i}\varepsilon$ in the α-stress field and another term with the eigenvalue $-1/2-\mathrm{i}\varepsilon$ in the β-stress field, has a contribution to the integral (49).

After using eqn (53), a simple derivation leads to the following path independent integral :

$$\begin{aligned} W &= \int_{\Gamma_2+\Gamma_1}\left[u_i\frac{\partial\sigma_{ij}}{\partial x} - \frac{\partial u_i}{\partial x}\sigma_{ij}\right]n_j\,\mathrm{d}s \\ &= -\pi\left(\frac{(1+\kappa_2)\kappa_1}{G_1} + \frac{(1+\kappa_1)\kappa_2}{G_2}\right)a_0\bar{a}_0(\tfrac{1}{2}-\mathrm{i}\varepsilon)(\tfrac{1}{2}+\mathrm{i}\varepsilon) \\ &= -\frac{1}{8}\left(\frac{(1+\kappa_2)\kappa_1}{G_1} + \frac{(1+\kappa_1)\kappa_2}{G_2}\right)(K_{1R}^2+K_{2R}^2). \end{aligned} \tag{61}$$

Also, after making integral by part and using the properties of ESF (Chen, 1985), eqn (61) can be rewritten in the form

$$\begin{aligned} J_R = \frac{W}{2} &= \int_{\Gamma_2+\Gamma_1}\left[U\,\mathrm{d}y - \frac{\partial u_i}{\partial x}\sigma_{ij}n_j\,\mathrm{d}s\right] \\ &= -\frac{1}{16}\left(\frac{(1+\kappa_2)\kappa_1}{G_1} + \frac{(1+\kappa_1)\kappa_2}{G_2}\right)(K_{1R}^2+K_{2R}^2) \end{aligned} \tag{62}$$

where U is the strain energy density. In eqn (62) the J_R integral possesses the same expression as in the crack problem. However, in the present case, the J_R integral takes a definitely negative value. This point was pointed out by (Chou and Wang, 1983) in the rigid line problem of the isotropic case.

(b) We take the α-stress field ($u_{i(\alpha)} = u_i$, $\sigma_{ij(\alpha)} = \sigma_{ij}$) as a physical stress field caused by some tractions acting on the boundary of the dissimilar body, and let the β-stress field be derived from the following complex potential

$$\begin{aligned} \phi_1(z) &= e_1 b_{-k} z^{-k+1/2-\mathrm{i}\varepsilon} && (z\in R_{1p}) \\ \omega_1(z) &= f_1 \bar{b}_{-k} z^{-k+1/2+\mathrm{i}\varepsilon} && (z\in R_{1p}) \\ \phi_2(z) &= e_2 b_{-k} z^{-k+1/2-\mathrm{i}\varepsilon} && (z\in R_{2p}) \\ \omega_2(z) &= f_2 \bar{b}_{-k} z^{-k+1/2+\mathrm{i}\varepsilon} && (z\in R_{2p}) \quad k = 1,2,\ldots \end{aligned} \tag{63}$$

After using the properties of EEF mentioned above, it is easy to see that only one pair, which consists of one term with the eigenvalue $k-1/2-i\varepsilon$ in the α-stress field and the β-stress field, has a contribution to the integral (49). A simple derivation leads to the following path independent integral

$$W = \int_{\Gamma_2+\Gamma_1} (u_i\sigma_{ij(\beta)} - u_{i(\beta)}\sigma_{ij})n_j\,ds$$
$$= -\pi\left(\frac{\kappa_1(1+\kappa_2)}{G_1} + \frac{\kappa_2(1+\kappa_1)}{G_2}\right)\mathrm{Re}\,[a_{k-1}\bar{b}_{-k}(k-\tfrac{1}{2}-i\varepsilon)]. \quad (64)$$

Also, some particular cases are cited below

$$W = -\pi\left(\frac{\kappa_1(1+\kappa_2)}{G_1} + \frac{\kappa_2(1+\kappa_1)}{G_2}\right)\mathrm{Re}\,[a_{k-1}] \quad (\text{if}\, b_{-k}(k-\tfrac{1}{2}+i\varepsilon) = 1, k = 1,2,\ldots)$$

$$W = -\pi\left(\frac{\kappa_1(1+\kappa_2)}{G_1} + \frac{\kappa_2(1+\kappa_1)}{G_2}\right)\mathrm{Im}\,[a_{k-1}] \quad (\text{if}\, b_{-k}(k-\tfrac{1}{2}+i\varepsilon) = i, k = 1,2,\ldots)$$

$$W = -\frac{\sqrt{\pi}}{2\sqrt{2}}\left(\frac{\kappa_1(1+\kappa_2)}{G_1} + \frac{\kappa_2(1+\kappa_1)}{G_2}\right)K_{1R} \quad (\text{if}\, k = 1, b_{-1} = 1)$$

$$W = \frac{\sqrt{\pi}}{2\sqrt{2}}\left(\frac{\kappa_1(1+\kappa_2)}{G_1} + \frac{\kappa_2(1+\kappa_1)}{G_2}\right)K_{2R} \quad (\text{if}\, k = 1, b_{-1} = i). \quad (65)$$

(c) We take the α-stress field ($u_{i(\alpha)} = u_i, \sigma_{ij(\alpha)} = \sigma_{ij}$) as a physical stress field caused by some tractions acting on the boundary of the dissimilar body, and let the β-stress field be derived from the following complex potential

$$\begin{aligned}
\phi_1(z) &= g_1 d_{-k} z^{-k} \quad (z \in R_{1p}) \\
\omega_1(z) &= h_1 \bar{d}_{-k} z^{-k} \quad (z \in R_{1p}) \\
\phi_2(z) &= g_2 d_{-k} z^{-k} \quad (z \in R_{2p}) \\
\omega_2(z) &= h_2 \bar{d}_{-k} z^{-k} \quad (z \in R_{2p}) \quad k = 1,2,\ldots
\end{aligned} \quad (66)$$

After using the properties of EEF mentioned above, it is easy to see that only one pair, which consists of one term with the eigenvalue k in the α-stress field and the β-stress field, has a contribution to the integral (49). A simple derivation leads to the following path independent integral

$$W = \int_{\Gamma_2+\Gamma_1} (u_i\sigma_{ij(\beta)} - u_{i(\beta)}\sigma_{ij})n_j\,ds$$
$$= k\pi\left(\frac{\kappa_1(1+\kappa_2)}{G_1} + \frac{\kappa_2(1+\kappa_1)}{G_2}\right)\mathrm{Re}\,[c_k\bar{d}_{-k}]. \quad (67)$$

Also, some particular cases are cited below

$$W = k\pi\left(\frac{\kappa_1(1+\kappa_2)}{G_1} + \frac{\kappa_2(1+\kappa_1)}{G_2}\right)\mathrm{Re}\,[c_k] \quad (\text{if}\, d_{-k} = 1, k = 1,2,\ldots)$$

$$W = k\pi\left(\frac{\kappa_1(1+\kappa_2)}{G_1} + \frac{\kappa_2(1+\kappa_1)}{G_2}\right)\mathrm{Im}\,[c_k] \quad (\text{if}\, d_{-k} = i, k = 1,2,\ldots). \quad (68)$$

4. WEIGHT FUNCTION

The pioneer investigations for weight function in crack problem were completed by Bueckner (1970) and Rice (1972). The higher order weight function formulation in the crack problem was suggested by Chen (1985), and Sham (1989). Also, the weight function in the interface crack problem was analyzed by Gao (1991). Clearly, a similar idea can be used to formulate the weight function in the problem. It is pointed out that the weight function in a rigid line problem likes the Green's function in the solution of Laplace's equation (Courant and Hilbert, 1962). Using the properties of EEF, the weight function formulation is also possible in the investigated problem. For a simplifying statement, the formulation is limited to the traction boundary problem along the outer boundary.

Below, the α-stress field is chosen as the physical stress field, and the β-stress field is derived from the following complex potentials:

$$\begin{aligned} \phi_{1s}(z) &= e_1 b_{-1} z^{-1/2-i\varepsilon} \quad (z \in R_{1p}) \\ \omega_{1s}(z) &= f_1 \bar{b}_{-1} z^{-1/2+i\varepsilon} \quad (z \in R_{1p}) \\ \phi_{2s}(z) &= e_2 b_{-1} z^{-1/2-i\varepsilon} \quad (z \in R_{2p}) \\ \omega_{2s}(z) &= f_2 \bar{b}_{-1} z^{-1/2+i\varepsilon} \quad (z \in R_{2p}). \end{aligned} \tag{69}$$

From eqns (63)–(65), we have the following result:

$$W = \int_{\Gamma_2+\Gamma_1} (u_i \sigma_{ij(\beta)} - u_{i(\beta)} \sigma_{ij}) n_j \, ds$$

$$= \begin{cases} -\dfrac{\sqrt{\pi}}{2\sqrt{2}} \left(\dfrac{\kappa_1(1+\kappa_2)}{G_1} + \dfrac{\kappa_2(1+\kappa_1)}{G_2} \right) K_{1R} & (\text{if } k = 1, b_{-1} = 1) \\ \dfrac{\sqrt{\pi}}{2\sqrt{2}} \left(\dfrac{\kappa_1(1+\kappa_2)}{G_1} + \dfrac{\kappa_2(1+\kappa_1)}{G_2} \right) K_{2R} & (\text{if } k = 1, b_{-1} = i). \end{cases} \tag{70}$$

For definiteness, the integration paths Γ_2 and Γ_1 are taken along the outer boundary of the body in Fig. 2. From eqn (70) we see that, in order to obtain the stress singularity coefficients

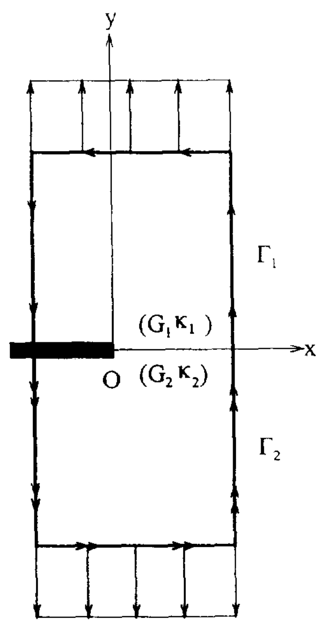

Fig. 2. An example of a bonded plate with a rigid line.

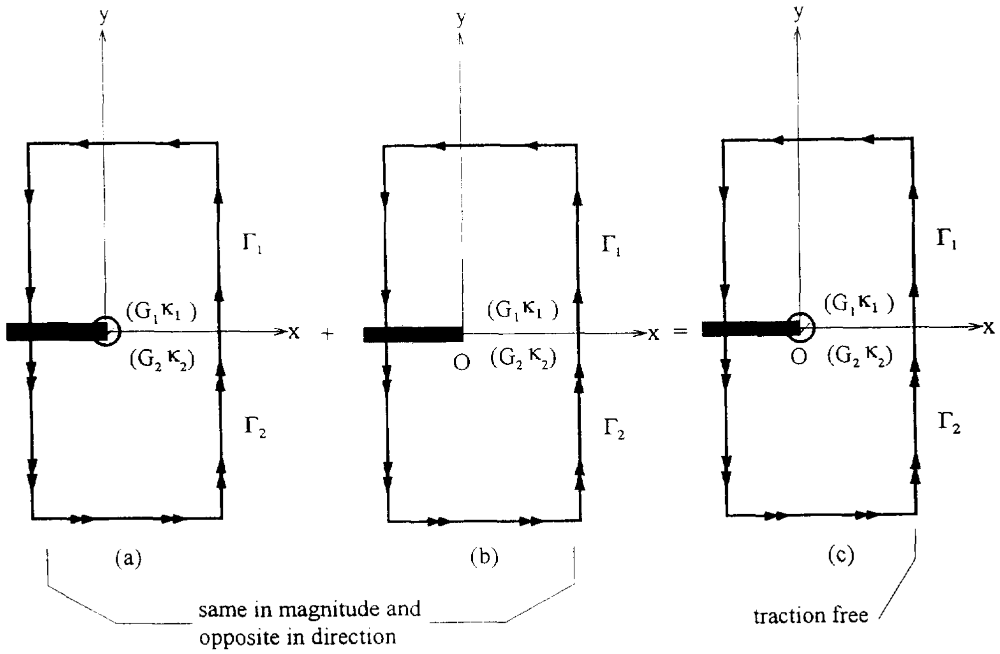

Fig. 3. A model for formulating the weight function approach.

at the crack tip, one has to know not only the tractions σ_{ij} but also the displacements u_i along the boundaries Γ_2 and Γ_1. Therefore, it is necessary to solve the traction boundary value problem. In addition, if there are many traction boundary value problems, one has to solve as many problems. The merit of the weight function is that, once a particular boundary value problem is solved, all the boundary value problems with the same geometry can be solved immediately. This idea is universal in mathematical physics. In the weight function method, the α-field is also of the physical stress field, and the β-field is derived from the complex potential with the following form:

$$\begin{aligned}
\phi_1(z) &= \phi_{1s}(z)+\phi_{1r}(z) \quad (z\in R_{1p}) \\
\omega_1(z) &= \omega_{1s}(z)+\omega_{1r}(z) \quad (z\in R_{1p}) \\
\phi_2(z) &= \phi_{2s}(z)+\phi_{2r}(z) \quad (z\in R_{2p}) \\
\omega_2(z) &= \omega_{2s}(z)+\omega_{2r}(z) \quad (z\in R_{2p}).
\end{aligned} \tag{71}$$

Two parts are involved in eqn (71), and the corresponding loading conditions are shown in Fig. 3(a,b), respectively. Note that the eigenvalue in eqn (69) is $-1/2-i\varepsilon$ and its real part is negative. Therefore, the stress field derived from the complex potential $\phi_{1s}(z)$, $\omega_{1s}(z)$ $(z\in R_{1p})$ and $\phi_{2s}(z)$, $\omega_{2s}(z)$ $(z\in R_{2p})$ is a singular stress field in the sense of unbounded displacement in the vicinity of the line tip. In addition, we can let the tractions along the Γ_2 and Γ_1 in Fig. 3(b) caused by $\phi_{1r}(z)$, $\omega_{1r}(z)$ $(z\in R_{1p})$ and $\phi_{2r}(z)$, $\omega_{2r}(z)$ $(z\in R_{2p})$ be opposite to those caused by $\phi_{1s}(z)$, $\omega_{1s}(z)$ $(z\in R_{1p})$ and $\phi_{2s}(z)$, $\omega_{2s}(z)$ $(z\in R_{2p})$. The superposition of two loading cases makes the β-stress field shown by Fig. 3(c). Clearly, the boundary value problem shown by Fig. 3(b) is a usual problem, and can be solved by any numerical method, for example, the finite element method. After considering the property of EEF, eqn (70) is still valid in this case. Since the traction $\sigma_{ij(\beta)}n_j$ vanishes along the contours Γ_2 and Γ_1 eqn (70) is reduced to the following equality:

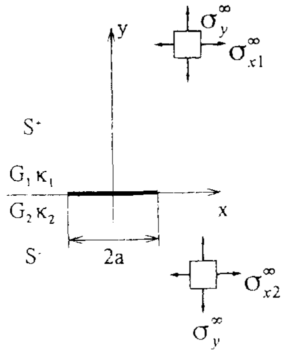

Fig. 4. A rigid line embedded in dissimilar media.

$$W = \int_{\Gamma_2+\Gamma_\varepsilon} (-u_{i(\beta)}\sigma_{ij}n_j)\,\mathrm{d}s$$

$$= \begin{cases} -\dfrac{\sqrt{\pi}}{2\sqrt{2}}\left(\dfrac{\kappa_1(1+\kappa_2)}{G_1}+\dfrac{\kappa_2(1+\kappa_1)}{G_2}\right)K_{1R} & (\text{if } k=1, b_{-1}=1) \\ \dfrac{\sqrt{\pi}}{2\sqrt{2}}\left(\dfrac{\kappa_1(1+\kappa_2)}{G_1}+\dfrac{\kappa_2(1+\kappa_1)}{G_2}\right)K_{2R} & (\text{if } k=1, b_{-1}=i). \end{cases} \tag{72}$$

In eqn (72), $u_{i(\beta)}$ shows the displacement along the boundaries Γ_1 and Γ_2 in Fig. 3(c), and serves as the weight function mentioned above. From the above analysis we see that, once the weight function is obtained, the stress singularity coefficients at the line tip for all the boundary value problems with the same geometry can be evaluated immediately. In fracture analysis, the original idea of the weight function was proposed by Bueckner (1970, 1973) and Rice (1972).

Obviously, the higher order weight functions in the interface crack problem can be formulated in a similar manner.

5. A RIGID LINE BETWEEN DISSIMILAR MEDIA

In this section we consider the two-dimensional elastic problem of a rigid line lying along the interface of two bonded dissimilar half-planes (Fig. 4). Two dissimilar half-planes are bonded together along the x-axis except for the interval $-a \leqslant x \leqslant a$ where a rigid line is embedded. We suppose the upper (the lower) half-plane S^+ (S^-) is occupied by a medium with elastic coefficients G_1 and κ_1(G_2 and κ_2), respectively. Since the rigid line is embedded, the elastic medium at the place, $y = 0^\pm$, $|x| < a$, cannot be deformed. For brevity, only normal mode is considered in the following analysis. Then it would seem that the following conditions must be satisfied on $y = 0$:

$$u_1 + iv_1 = 0 \quad (y = 0^+, |x| \leqslant a)$$
$$u_2 + iv_2 = 0 \quad (y = 0^-, |x| \leqslant a) \tag{73}$$

and

$$u_1 + iv_1 = u_2 + iv_2 \quad (y = 0, |x| \geqslant a)$$
$$-Y_1 + iX_2 = -Y_2 + iX_2 \quad (y = 0, |x| \geqslant a). \tag{74}$$

In the analysis, the complex potentials $\phi_1(z)$ and $\psi_1(z)$ are defined in the upper half-plane, $\phi_2(z)$, $\psi_2(z)$ in the lower half-plane, respectively.

To perform the derivation, we define the following functions

$$\eta_1(z) = z\overline{\phi'_2}(z) + \overline{\psi_2}(z) \quad (z \in S^+)$$
$$\eta_2(z) = z\overline{\phi'_1}(z) + \overline{\psi_1}(z) \quad (z \in S^-). \tag{75}$$

The continuation condition of the forces and the displacements along $y = 0$, $|x| \geqslant a$ will lead to

$$\phi_1^+(x) + \eta_2^-(x) = \phi_2^-(x) + \eta_1^+(x) \quad (|x| \geqslant a)$$
$$G_2[\kappa_1\phi_1^+(x) - \eta_2^-(x)] = G_1[\kappa_2\phi_2^-(x) - \eta_1^+(x)] \quad (|x| \geqslant a) \tag{76}$$

or in an alternative form

$$\phi_1^+(x) - \eta_1^+(x) = \phi_2^-(x) - \eta_2^-(x) \quad (|x| \geqslant a)$$
$$G_2\kappa_1\phi_1^+(x) + G_1\eta_1^+(x) = G_1\kappa_2\phi_2^-(x) + G_2\eta_2^-(x) \quad (|x| \geqslant a). \tag{77}$$

This means, for example, the analytic function $\phi_2(z) - \eta_2(z)$ defined in the lower half-plane is a continuation of the analytic function $\phi_1(z) - \eta_1(z)$ defined in the upper half-plane. Thus, we can put

$$\phi_1(z) - \eta_1(z) = \delta(z) \quad (z \in S^+) \tag{78a}$$
$$\phi_2(z) - \eta_2(z) = \delta(z) \quad (z \in S^-) \tag{78b}$$

$$G_2\kappa_1\phi_1(z) + G_1\eta_1(z) = \theta(z) \quad (z \in S^+) \tag{79a}$$
$$G_1\kappa_2\phi_2(z) + G_2\eta_2(z) = \theta(z) \quad (z \in S^-) \tag{79b}$$

where $\delta(z)$ and $\theta(z)$ are two holomorphic functions in the whole plane cut along $(-a, a)$.

After solving eqns (78a) and (79a), (78b) and (79b), we find

$$\phi_1(z) = \frac{\theta(z) + G_1\delta(z)}{G_1 + G_2\kappa_1}, \quad \eta_1(z) = \frac{\theta(z) - G_2\kappa_1\delta(z)}{G_1 + G_2\kappa_1} \quad (z \in S^+) \tag{80}$$

$$\phi_2(z) = \frac{\theta(z) + G_2\delta(z)}{G_2 + G_1\kappa_2}, \quad \eta_2(z) = \frac{\theta(z) - G_1\kappa_2\delta(z)}{G_2 + G_1\kappa_2} \quad (z \in S^-). \tag{81}$$

From the condition of stresses applied at infinity (Muskhelishvili, 1953), we find the following asymptotic behaviour

$$\phi_1(z) = \Gamma_1 z + \ldots \tag{82a}$$
$$\psi_1(z) = \Gamma'_1 z + \ldots \tag{82b}$$
$$\eta_1(z) = (\Gamma_2 + \Gamma'_2)z + \ldots \tag{82c}$$
$$\delta(z) = [\Gamma_1 - (\Gamma_2 + \Gamma'_2)]z + \ldots \tag{82d}$$
$$\theta(z) = [G_2\kappa_1\Gamma_1 + G_1(\Gamma_2 + \Gamma'_2)]z + \ldots \quad (z \in S^+) \tag{82e}$$

and

$$\phi_2(z) = \Gamma_2 z + \ldots \tag{83a}$$
$$\psi_2(z) = \Gamma'_2 z + \ldots \tag{83b}$$
$$\eta_2(z) = (\Gamma_1 + \Gamma'_1)z + \ldots \tag{83c}$$

$$\delta(z) = [\Gamma_2 - (\Gamma_1 + \Gamma_1')]z + \dots \tag{83d}$$

$$\theta(z) = [G_1\kappa_2\Gamma_2 + G_2(\Gamma_1 + \Gamma_1')]z + \dots \quad (z \in S^-) \tag{83e}$$

where

$$\Gamma_1 = \frac{\sigma_y' + \sigma_{x1}'}{4} \quad \Gamma_1' = \frac{\sigma_y' - \sigma_{x1}'}{2} \tag{84}$$

$$\Gamma_2 = \frac{\sigma_y' + \sigma_{x2}'}{4} \quad \Gamma_2' = \frac{\sigma_y' - \sigma_{x2}'}{4}. \tag{85}$$

Since $\delta(z)(z \in S^+)$ is a continuation of $\delta(z)(z \in S^-)$, from eqns (82d) and (83d) we find

$$\Gamma_1 - \Gamma_2 - \Gamma_2' = \Gamma_2 - \Gamma_1 - \Gamma_1'. \tag{86}$$

Similarly, from eqns (82e) and (83e) it follows

$$G_2\kappa_1\Gamma_1 - G_1(\Gamma_2 + \Gamma_2') = G_1\kappa_2\Gamma_2 + G_2(\Gamma_1 + \Gamma_1'). \tag{87}$$

It is proved that eqn (86) is an identity, and eqn (87) will lead to the well known compatibility condition

$$G_1(1+\kappa_2)\sigma_{x2}' = G_2(1+\kappa_1)\sigma_{x1}' + [G_1(3-\kappa_2) - G_2(3-\kappa_1)]\sigma_y'. \tag{88}$$

From eqns (5) and (73), we get the following conditions along the rigid line

$$\kappa_1\phi_1^+(x) - \eta_2^-(x) = 0 \quad (|x| \leqslant a) \tag{89}$$

$$\kappa_2\phi_2^-(x) - \eta_1^+(x) = 0 \quad (|x| \leqslant a).$$

Substituting eqns (80) and (81) into eqn (89) yields

$$\frac{G_1\kappa_1}{G_1 + G_2\kappa_1}\delta^+(x) + \frac{G_1\kappa_2}{G_2 + G_1\kappa_1}\delta^-(x) + \frac{\kappa_1}{G_1 + G_2\kappa_1}\theta^+(x) - \frac{1}{G_2 + G_1\kappa_2}\theta^-(x) = 0 \quad (|x| \leqslant a) \tag{90a}$$

$$\frac{G_2\kappa_1}{G_1 + G_2\kappa_1}\delta^+(x) + \frac{G_2\kappa_2}{G_2 + G_1\kappa_2}\delta^-(x) - \frac{1}{G_1 + G_2\kappa_1}\theta^+(x) + \frac{\kappa_2}{G_2 + G_1\kappa_2}\theta^-(x) = 0 \quad (|x| \leqslant a). \tag{90b}$$

From the above equation $[G_2 * \text{eqn (90a)} - G_1 * \text{eqn (90b)}]$, we obtain

$$\theta^+(x) - \theta^-(x) = 0 \quad (|x| \leqslant a). \tag{91}$$

This proves that the function $\theta(z)$ is holomorphic in the entire plane. And, from eqns (82e) and (83e) the function $\theta(z)$ takes the form

$$\theta(z) = ez, \quad e = G_2\kappa_1\Gamma_1 - G_1(\Gamma_2 + \Gamma_2') = G_1\kappa_2\Gamma_2 + G_2(\Gamma_1 + \Gamma_1'). \tag{92}$$

Substituting eqn (92) into eqn (90a) yields the following Hilbert problem (Muskhelishvili, 1953)

$$\delta^+(x) = g\delta^-(x) + f(x) \tag{93}$$

where

$$g = -\frac{\kappa_2(G_1+G_2\kappa_1)}{\kappa_1(G_2+G_1\kappa_2)}$$

$$f(x) = hx, \quad h = \frac{1-\kappa_1\kappa_2}{\kappa_1(G_2+G_1\kappa_2)}[G_2\kappa_1\Gamma_1 + G_1(\Gamma_2+\Gamma_2')]. \tag{94}$$

The Hilbert problem has a solution as follows (Muskhelishvili, 1953):

$$\delta(z) = \frac{h}{1-g}\{z - X_*(z)\} + cX_*(z) \tag{95}$$

where

$$X_*(z) = (z+a)\left(\frac{z-a}{z+a}\right)^{1/2-i\varepsilon}$$

$$\varepsilon = \frac{1}{2\pi}\log\left(\frac{\kappa_2(G_1+G_2\kappa_1)}{\kappa_1(G_2+G_1\kappa_2)}\right)$$

$$c = \Gamma_1 - (\Gamma_2+\Gamma_2') \tag{96}$$

and the constant c is obtained from the asymptotic expression shown by eqn (82d).

Finally, from eqns (80), (92) and (95), we get

$$\phi_1(z) = \frac{1}{G_1+G_2\kappa_1}\left\{ez + G_1\left(\frac{h}{1-g}[z - X_*(z)]\right) + cX_*(z)\right\} \quad (z\in S^+) \tag{97}$$

and we can rewrite $\phi_1(z)$ in the form

$$\phi_1(z) = \phi_{1a}(z) + \phi_{1b}(z) \tag{98}$$

with

$$\phi_{1a}(z) = \frac{1}{G_1+G_2\kappa_1}\left(c - \frac{h}{1-g}G_1\right)X_*(z)$$

$$\phi_{1b}(z) = \frac{1}{G_1+G_2\kappa_1}\left(e + \frac{h}{1-g}G_1\right)z \tag{99}$$

Clearly, only the portion of $\phi_{1a}(z)$ has a contribution to the stress singularity coefficient. Thus from eqn (2) we can let

$$K_{1R} - iK_{2R} = 2\sqrt{2\pi}\sqrt{\frac{G_2\kappa_1}{G_1\kappa_2}}e^{\pi\varepsilon}\lim_{z\to a}(z-a)^{1/2+i\varepsilon}\phi'_{1a}(z) \tag{100}$$

and obtain

$$K_{1R}-\mathrm{i}K_{2R} = 2\sqrt{\pi a}\sqrt{\frac{G_2\kappa_1}{G_1\kappa_2}}(1-2\mathrm{i}\varepsilon)(2a)^{\mathrm{i}\varepsilon}\frac{1}{G_1+G_2\kappa_1}\left(c-\frac{h}{1-g}G_1\right) \tag{101}$$

where the constants ε, c, h, g have been shown in eqns (94) and (96).

The loading condition at infinity may be decomposed into two particular cases.

(a) If $\sigma_y^\infty = 0$, we choose σ_{x1}^∞ as independent, and from eqn (88) it follows

$$\sigma_{x2}^\infty = \sigma_{x1}^\infty\frac{G_2(1+\kappa_1)}{G_1(1+\kappa_2)}. \tag{102}$$

In this case, from equation (101) we find

$$K_{1R}-\mathrm{i}K_{2R} = \sqrt{\frac{G_2\kappa_1}{G_1\kappa_2}}\mathrm{e}^{\pi\varepsilon}(1-2\mathrm{i}\varepsilon)(2a)^{\mathrm{i}\varepsilon}\sqrt{\pi a}H \tag{103}$$

where

$$H = \frac{(1+\kappa_1)(G_2+G_1\kappa_2)}{2[\kappa_1(G_2+G_1\kappa_2)+\kappa_2(G_1+G_2\kappa_1)]}\sigma_{x1}^\infty. \tag{104}$$

In the isotropic case ($G_1 = G_2 = G, \kappa_1 = \kappa_2 = \kappa$), from eqns (103) and (104) we obtain the well known result $K_{1R}-\mathrm{i}K_{2R} = (\kappa+1)\sigma_x^\infty\sqrt{\pi a}/(4\kappa)$.

(b) If $\sigma_{x1}^\infty = 0$, we choose σ_y^∞ as independent, and from eqn (88) it follows

$$\sigma_{x2}^\infty = \frac{1}{G_1(1+\kappa_2)}[G_1(3-\kappa_2)-G_2(3-\kappa_1)]\sigma_y^\infty. \tag{105}$$

In this case, eqn (103) is still used, and H becomes

$$H = \frac{(\kappa_1-3)(G_2+\kappa_2G_1)}{2[\kappa_1(G_2+G_1\kappa_2)+\kappa_2(G_1+G_2\kappa_1)]}\sigma_y^\infty. \tag{106}$$

In the isotropic case ($G_1 = G_2 = G, \kappa_1 = \kappa_2 = \kappa$), from eqns (103) and (106), we obtain the well known result $K_{1R}-\mathrm{i}K_{2R} = (\kappa-3)\sigma_y^\infty\sqrt{\pi a}/(4\kappa)$.

An earlier derivation for the interface rigid line problem has been carried out by Ballarini (1990). However, some particular points can be found from our study. Since the complex potentials $\phi_1(z)$, $\psi_1(z)$ (for the upper plane) and $\phi_2(z)$, $\psi_2(z)$ (for the lower plane) have been derived in an explicit form, the whole stress and displacement field can be evaluated immediately. Secondly, it is more natural to define the stress singularity coefficient though the expression of the complex potential. Also, the definition shown by eqn (60) can be easily reduced to the isotropic case, and can be compared with the definition for evaluating the stress intensity factor in the crack problem case.

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